Strong convergence (in Norm)

A seq $\{x_n\}$ in a normed X is said to be strongly convergent if there is an $x \in X$ $\ni$

$$\lim_{n \to \infty} \|x_n - x\| = 0$$

$$\lim_{n \to \infty} x_n = x$$

x is called strong limit of $\{x_n\}$ and we say that $\{x_n\}$ converges strongly to x.

Weak convergence

A seq $\{x_n\}$ in a normed space X is said to be weakly convergent if there is an $x \in X$ $\ni$ for every $f \in X'$ (collection of all bounded linear functional)

$$\lim_{n \to \infty} f(x_n) = f(x)$$

This element x is called weak limit of $\{x_n\}$ and we say that $\{x_n\}$ converges weakly to x.

$\Rightarrow$ Imp: weak convergence means convergence of the seq of scalar no $\{a_n\} = \{f(x_n)\}$ to a no $f(x)$ for every $f \in X'$

Weak convergence property

Let $\{x_n\}$ be a weakly convergent seq in a normed space X

Say $x_n \xrightarrow{w} x$

(a) The weak limit x of $\{x_n\}$ is unique

(b) every Subseq of $\{x_n\}$ converges weakly to x

(c) The seq $\|x_n\|$ is bounded

The
$\Rightarrow$

If x_0 is such that $f(x_0) = 0$ for all $f \in X'$ then $x_0 = 0$

uniform boundness

let T_n be a seq of bounded linear operator from a Banach space to a normed space then boundeness $\overset{\text{implies}}{\Leftarrow}$ uniform boundeness.

Theorem

strong and weak convergence

(a) strong convergence emplies weak convergence with the same limit

(b) The converse is not true

(b) If $\dim X < \infty$ then weak convergence $\Leftrightarrow$ strong convergence

Converg exp

Riesz theorem

Total set

A total set M in a

normed space X is a set whose span is dense in X that is

$\overline{\text{span } M} = X$

=> In every Hilbert space $H \neq \{0\}$ there exist a total orthonormal set

<u>Lema</u> (weak convergence)

In a normed space X we have $x_n \xrightarrow{w} x$ iff.

(a) The seq $\{\|x_n\|\}$ is bounded

(b) for every element f of a total subset $M \subset X$ we have

$$f(x_n) \to f(x)$$

The => In a Hilbert space $x_n \xrightarrow{w} x$ iff $(x_n, z) \to (x, z)$ for all z in the space.

=> Let X, Y be normed space and $B(X, Y)$ be the space of all bounded linear operator $T: X \to Y$ we have 3 type of seq

① T_n in $B(X, Y)$

② $\{T_n(x)\}$ seq of vector in Y for each $x \in X$

③ $\{f(T_n(x))\}$ seq of scalar for $f \in Y$

<u>Three type of convergence</u>

① Uniform operator convergence

A seq $\{T_n\}$ of operator in $B(X, Y)$ is said to be uniform operator converges if there exist an operator $T : X \to Y$ ∋

$$\|T_n - T\| \to 0 \quad as \quad n \to \infty$$

T is called u o limit of the seq $\{T_n\}$

② Strong operator convergence

A seq $\{T_n\}$ of operator in $B(X, Y)$ is said to be strongly operator convergent if there exist an operator $T : X \to Y$ ∋

$$\|T_n(x) - T(x)\| \to 0 \quad as \quad n \to \infty \quad x \in X$$

T is called the strong operator limit of the seq $\{T_n\}$

③ weak operator convergenc

$$\|f(T_n(x)) - f(Tx)\| \to 0 \quad x \in X, \; f \in Y'$$

Result

The u.o.c ⟹ Strong o. convergence but converse is not true

e.g. $\ell^2 : \{x = (x_1, x_2 \ldots) : \sum_{i}^{\infty} |x_i|^2 < \infty$

The ⟹ T_n is bounded and $\|T_n\| = 1$

T_n converge strong to zero operator O

strong mein hum vector use

Tn is not uniform convergent to zero operator

Result $\overline{Q}$

S.O. convergent $\Rightarrow$ weakly operator convergent but converse is not true

e.g $l^2 = \{ x = (x_1, x_2 ..) \mid \sum_{i=1}^{\infty} |x_i|^2 < \infty$

Tn is linear

Tn converg weakly to zero operator

Tn is bounded $\|Tn\| = 1$

Tn does not converges strongly.

v.v.i Banach fixed point

def

let X be non-empty set mappening $d : X - X \to R$ is said to metric and fallowing 4 property

v.v. complete

The space X is said to be complete iff every cauchy seq X convergence to a point in X

Contraction

Let (X, d) be a metric space. A map $T : X \to X$ is called contraction on X if there is a positive real no $\alpha < 1$ $\exists$ $x, y \in X$

every continuous function is contraction

$d(Tx, Ty) \leq \alpha d(x,y)$

e.g

$T(x) = \frac{1}{2}x$, $x \in R$

$|Tx - Ty| = \frac{1}{2}|x-y|$

Fixed point

let X be non empty set $T: X \to X$ be a maping. A point $x \in X$ is said to be fixed of T if $x = T(x)$

e.g

$T: R \to R$

$T(x) = \frac{1}{2}x$ {0 is fixed point}

e.g. $T: [1, \infty) \to [1, \infty)$ be difened

$T(x) = x + x^{-1}$ has no fixed point.

e.g $T: R \to R$

$T(x) = x+1$ no fixed pt

$T(x) = x^2$ 0, 1 fixed point

e.g let $T: R \to R$

$T(x) = x$ every real no is fixed

Problem of fixed point

① existence ② uniqueness