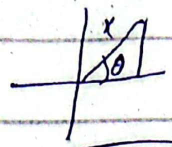
Hilbert spaces

$X \neq \phi$ {N, W, Sets, Matrix etc}

$R^2 \Rightarrow$



$d(x,y) = \sqrt{(x_1-y_1)+(x_2-y_2)}$

metric [$d: X \times X \to R^+ \cup [0]$

Vector $\begin{cases} +: X \times X \to Y \iff (x_1+x_2),(y_1+y_2) \in R^2 \\ \times: K \times X \to X \iff (\alpha x_1, \alpha x_2) \quad \alpha \in R \end{cases}$

Norm $\Rightarrow \|\cdot\|: X \to R^+ \cup [0]$

a.b $\Rightarrow$ prependicular

$X \neq \phi \quad \langle , \rangle : X \times X \to K$

$X(K) \quad :: \quad K = C$ or R

① $\langle x+y, z\rangle = \langle x,z\rangle + \langle y,z\rangle$

② $\langle \alpha x, y\rangle = \alpha \langle x,y\rangle$

③ $\langle x,y\rangle = \overline{\langle y,x\rangle}$

④ $\langle x,x\rangle \geq 0$ $\qquad A \cdot A = A^2 \geq 0$

Result of definition

① $K = R$, $\langle x,y\rangle = \overline{\langle y,x\rangle} = \langle y,x\rangle \because \bar{a} = a$

② $\langle \alpha_1 x_1 + \alpha_2 x_2, z\rangle = \langle \alpha_1 x_1, z\rangle + \langle \alpha_2 x_2, z\rangle$

$\langle \sum_{i=1}^{2} \alpha_i x_i, z\rangle = \alpha\langle x_1, z\rangle + \alpha_2 \langle x_2, z\rangle$

$\qquad = \sum_{i=1}^{2} \alpha_i \langle x_i, z\rangle$

③ $\langle x, \alpha y\rangle = \overline{\langle \alpha y, x\rangle} \Rightarrow \bar{\alpha}\langle y,x\rangle = \bar{\alpha}(x,y)$

④ $\langle x, \alpha_1 y_1 + \alpha_2 y_2\rangle = \bar{\alpha}_1 \langle x, y_1\rangle + \bar{\alpha}_2 \langle x, y_2\rangle$

Result v.v.v. Imp

Th. ① An inner product on X a norm on X

$$||x|| = \sqrt{\langle x, x \rangle} \qquad x \geq 0$$

② metric

$$d(x,y) = ||x-y|| = \sqrt{\langle x-y, x-y \rangle}$$

Inner = norm = metric = Topology

but converse not true

Sheirkpinh ~~Ato~~ Space is not metric

A discrete metric is not normed

l_p, $p \neq 2$, $C[a,]$ is not

Inner product space.

$a \cdot b = ab\cos\theta$

$|a \cdot b| = |ab \cos\theta|$

$|a \cdot b| = |a||b||\cos\theta| \qquad \because |\cos\theta| \leq 1$

$|a \cdot b| \leq |a||b|$

Identity

① parallelogram

$(a+b)^2 + (a-b)^2 = 2(a^2+b^2)$

$||x+y||^2 + ||x-y||^2 = 2[||x||^2 + ||y||^2]$

② $||x+y||^2 - ||x-y||^2 = 4\langle x, y \rangle$

Example of Inner product

R, $\langle x, y \rangle = xy$

C, $\langle x, y \rangle = x\bar{y}$

R^n $\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i$

C^n $\langle x, y \rangle = \sum_{i=1}^{n} x_i \bar{y_i}$

l^2 hilbert space

$\langle x, y \rangle = \sum_{i=1}^{\infty} x_i \bar{y_i}$

Some other example where are norm space but not inner product

① l^p space with $p \neq 2$ is not inner

$\{ (x_n) ; \sum_{n=1}^{\infty} |x_n|^p < \infty$

l^p is complete

② Space $C[a,b]$ is not inner product not a hilbert space

apollonius identity

$$\|z-x\|^2 + \|z-y\|^2 = \frac{1}{2}\|x-y\|^2 + 2\|z - \frac{1}{2}(x+y)\|^2$$

v.v.imp ⑨ set of orthogonal vector then they are L.I

Continuity of inner product

If in an inner product

$x_n \to x$, $y_n \to y$ then

$(x_n, y_n) \to (x, y)$

orthogonal complements and direct sums

① minimize vector

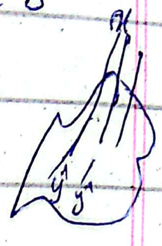
② annihilator

③ Direct sum

④ Projection

Minimizing vector theorem

A subset $M \subset X$ the distance δ from an element $x \in X$ to a is defined



$\delta = \inf \{ d(x, \tilde{y}),\ \tilde{y} \in M)$

$$d(x, y) = \delta$$

then y is minimize vector

But in norm space

$X =$

$$\delta = \inf d(x, \tilde{y}) \quad \forall\ \tilde{y} \in M$$

$$= \inf_{\tilde{y} \in M} \|x - \tilde{y}\|$$

If $y \in M$ then $\|x - y\| = \delta$

then y is closest point of M

e.g

$x = \pi$ and $M =$ ration

$3.14, \frac{22}{7}$ M

e.g e^x $M = P(x)$ but e^x is not polynomial

$1 + x + \frac{x^2}{2}$

Segment

$\boxed{x - y}$ $\Rightarrow \alpha x + (1 - \alpha) y$

$\alpha \in \mathbb{R} \quad 0 \leq \alpha \leq 1$

Convex

circle convex | not convex
Squar convex |

$M \subset X$ convex if $\forall\, x, y \in M$ $\alpha \in [0,1]$

$\alpha x + (1-\alpha) y \in M$

Theorem (Minimize vector)

Let X be an inner product space and $M \neq \phi$ a convex set which is complete (in the metric induced by inner product). Then for every given $x \in X$ their exist a unique $y \in M$

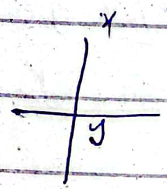
$\exists \quad \delta = \inf_{\tilde{y} \in M} \|x - \tilde{y}\| = \|x - y\|$

v.v
$\Rightarrow$ orthogonality

$M \neq \phi$, complete, convex

$d(x, y) = \delta$

$x - y \perp M$



The $\Rightarrow$ Let M be a complete subspace Y and $x \in X$ fixed then $z = x - y$ is orthogonal to y

direct sum

A vector space X is said to be direct sum of two subspace Y and Z

Hilbert $= R, R^n$
and dual space is itself
~~dttt~~

$X = Y \oplus Z$ $\qquad x \in X$

$x = y + z$ $\qquad y \in Y, z \in Z$

The

(1) let Y be any closed subspace of a hilbert space H then

$H = Y \oplus Z \qquad Z = Y^{\perp}$

$Y^{\perp} = \{x \in H, x \perp y\}$

(2) $R^2 : x \in R^2 \Rightarrow (x_1 i + x_2 j) = x$

$x \times x$ $\qquad Y \cap Y^{\perp} = \{0\}$ v.v. Imp

v.v.v
$\Rightarrow$ ~~or~~ orthogonal projection

$H = Y \oplus Y^{\perp}$

$P : H \to Y \Rightarrow$ closed

$x \in H \qquad x = y + z$

$P(x) = P(y+z) = y$

Property

(1) linear (2) bounded (3) bounded linear operator

(4) onto (5) idempotent

(6) $P|_Y = I_Y$ (7) $N(P) = Y^{\perp}$

Functional on Hilbert space

what is Hilbert space v.v. Imp

Hilbert space is a complete ~~normed~~ inner product space.

Riesz's Theorem V.V.V. Imp

Every bounded linear functional f on a hilbert space H can be

$f(x) = \langle x, z \rangle$ z depend on f $\|z\| = \|f\|$